NEW YORK UNIVERSITY
Institute of Mathematical Sciences
Division of Electromagnetic Research

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The Effect of Discontinuities of Dielectric Constant on Electrostatic Fields near Conductors

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Abstract

This paper contains an investigation of the effect of a local discontinuity of dielectric constant upon the singularity in field strength in the neighborhood of a geometrical singularity of a conductor. The work is compared to the electrostatic case. We first consider the case of a wedge-shaped conductor in the presence of a plane surface of discontinuity passing through the vertex of the wedge. For this problem the dependence of the singularity on the various parameters is studied in detail, and in addition the Green's function is derived. It is found that a) when the wedge is symmetric with respect to the interface the singularity is unaffected by the presence of a discontinuity. It is also noted that b) in the symmetric case the Green's function can be expressed, by the method of images, in terms of the Green's function for a conducting wedge in free space. The result b) is then extended to the case of an arbitrary symmetric distribution of conducting material. The corresponding extension of the result a) is then deduced.

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1. Introduction

It is well known that electric fields tend to become singular near sharp edges or peaks in conducting bodies. Bouwkamp [1], Meixner [2], Maue [3], Jones [4] and others have discussed the character of the singularities and the restrictions which must be imposed on them to insure uniqueness and physical significance, in the case of locally wedge-shaped conductors in homogeneous media. The purpose of the present work is to investigate the effect of a discontinuity of medium on the singularity. The investigation is confined to the electrostatic case, but it is hoped that the information obtained may prove of some value also in the electromagnetic case, since both cases usually present similar behavior at singularities. In addition, the electrostatic results obtained may be of some intrinsic interest. Some of these results are independent of the form of the conductor.

This investigation originated in an attempt to explain the phenomena which occur under certain conditions when irises in waveguides are covered with transparent plastics. It is observed that the plastic tends to become charred and pitted in the neighborhood of the edges of the screen*. Since this phenomenon could be ascribed to the high field strength at an edge, it seemed to be of interest to determine how and whether the singularity of the field differs from that occurring when the aperture is left open.

We proceed to outline the course of the investigation. In Section 2 we study the behavior of the singular part of the electrostatic solution in the neighborhood of the edge of a wedge-shaped conductor in the presence of two media with two different dielectric constants, ε_1 and ε_2. This configuration is further explained by Figure 1.

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* Oral communication from Dr. Morris Ettenberg.

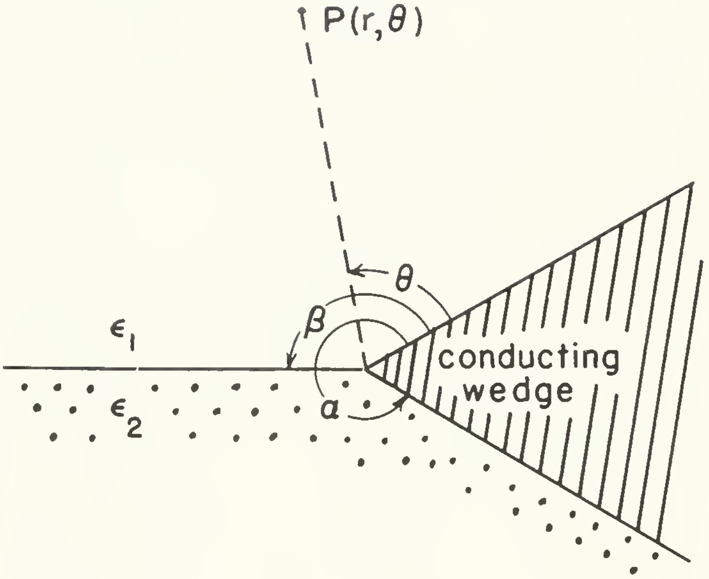

Figure 1

It is assumed that the leading term in the potential is of the form r^λ, and an equation for λ is obtained. In general λ depends upon the quantities α/β and $\varepsilon_1/\varepsilon_2$. However, it is found that when $\beta = \alpha/2$ then $\lambda = \pi/\alpha$, independently of ε_1 and ε_2. Thus when the conducting wedge is placed symmetrically with respect to the interface, the discontinuity of dielectric constant has no effect on the local functional form of the potential or field.

Interesting situations occur when $\alpha = 3\pi/2$ and $\beta = \pi/2$ or $\beta = \pi$. These cases arise when an aperture in a conducting screen of the cross section indicated in Figures 2a and 2b is closed by a sheet of dielectric material ε_2, the dielectric in the remaining space being ε_1, with $\varepsilon_1 < \varepsilon_2$.

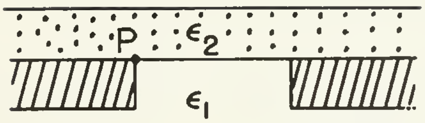

Figure 2a

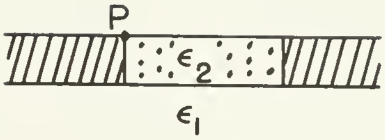

Figure 2b

In the case of Figure 2a it turns out that the field strength near P is of a higher order of magnitude than would be found near P if $\varepsilon_2 = \varepsilon_1$. In the case of Figure 2b the reverse if the case.

We conclude Section 2 by giving variational formulas which allow the calculation of λ for small deviations from symmetry, or homogeneity of medium.

Section 3 is devoted to the determination of the two-dimensional static Green's function for the configuration of Figure 1. A finite Fourier transformation of a special kind is employed for this purpose. When the interface is symmetrical it turns out that the resulting solution can be expressed by the method of images, with the line source replaced by the free-space Green's function for the wedge.

In Section 4 the final result of Section 3 is generalized to three dimensions, and to arbitrary symmetric screens, in the possible presence of a symmetric distribution of other conductors. The new result is a generalization of a well-known result of Lord Kelvin. It is useful in a qualitative sense since by comparing the singular part of the Green's function with the singularity occurring in a homogeneous medium, one can show there is no change, despite the fact that the situation in the homogeneous medium is not known analytically, but only experimentally. The explanation of this fact is discussed in Section 5.

Our general conclusions are summarized in Section 6.

2. Local behavior near the tip of a wedge

Figure 3 represents in cross section the conditions near the edge of a wedge-shaped conductor. The exterior angle of the wedge is α. The vertex of the wedge is taken as the origin of polar coordinates r, θ. The ray $\theta = \beta$ represents the interface between medium I, whose dielectric constant is ε_1, and medium II, with dielectric constant ε_2. Let $u(r,\theta)$ be the electrostatic potential due to some unspecified excitation.

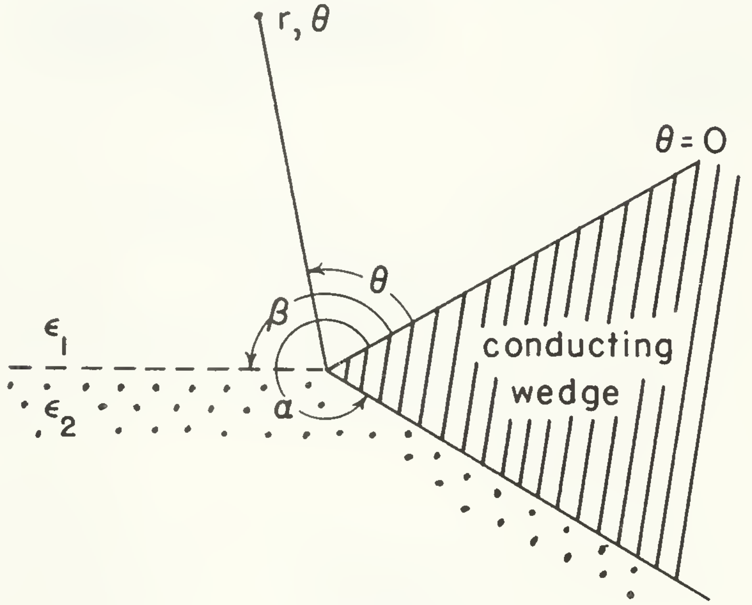

Figure 3

In general the following conditions must be satisfied by an electrostatic potential:

a) u obeys Laplace's equation;

b) u vanishes on conductors;

c) u must be continuous at the interface between two different media;

d) let $\frac{\partial}{\partial n}$ denote differentiation normal to the interface between two media. Then $\varepsilon \frac{\partial u}{\partial n}$ must be continuous ($\varepsilon = \varepsilon_1, \varepsilon_2$ in the first and second mediums respectively).

In the present configuration these conditions may be restated as follows:

(2.1) $$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0 ;$$

(2.2) $$u(r,0) = u(r,\alpha) = 0 ;$$

(2.3) $$\lim_{\delta \to 0} u(r,\beta-\delta) = \lim_{\delta \to 0} u(r,\beta+\delta) ;$$

(2.4) $$\lim_{\delta \to 0} \varepsilon_1 \frac{\partial u}{\partial \theta} (r,\beta-\delta) = \lim_{\delta \to 0} \varepsilon_2 \frac{\partial u}{\partial \theta} (r,\beta+\delta) .$$

We now assume that for sufficiently small r, the potential $u(r,\theta)$ may be represented in the form

(2.5) $$u(r,\theta) \cong r^{\lambda_1} f(\theta) ,$$

where λ_1 is a constant to be determined. Then it follows from d) that $f(\theta)$ is of the form

(2.6) $$f(\theta) = A \cos \lambda_1 \theta + B \sin \lambda_1 \theta .$$

To meet condition (2.2) we set

(2.7) $$\begin{aligned} f(\theta) &= A \sin \lambda_1 \theta ; && \beta > \theta \geq 0 , \\ f(\theta) &= B \sin \lambda_1 (\alpha-\theta) ; && \alpha \geq \theta > \beta . \end{aligned}$$

Then condition (2.3) implies

(2.8) $$B = A \frac{\sin \lambda_1 \beta}{\sin \lambda_1 (\alpha-\beta)} .$$

Finally, (2.4) implies

(2.9) $$\varepsilon_1 A \lambda_1 \cos \lambda_1 \beta = \varepsilon_2 A \lambda_1 \frac{\sin \lambda_1 \beta}{\sin \lambda_1 (\alpha-\beta)} \cos \lambda_1 (\alpha-\beta) .$$

The last equation can be used for the determination of λ. Setting $\gamma = \alpha-\beta$, we have

(2.9') $$\varepsilon_1 \cos \lambda_1 \beta \sin \lambda_1 \gamma + \varepsilon_2 \sin \lambda_1 \beta \cos \lambda_1 \gamma = 0 .$$

This equation may be transformed, by trigonometric manipulation, into the more convenient form

(2.10) $$\sin \lambda_1(\beta+\gamma) = \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2} \sin \lambda_1(\beta-\gamma) ,$$

where $\alpha = \beta + \gamma$.

Consider first the special case $\varepsilon_1 = \varepsilon_2$. Then we have $\sin \lambda_1(\beta+\gamma) = 0$, and thus there are infinitely many roots

$$\lambda_1 = \frac{n\pi}{\alpha}, \quad n = 0, \pm 1, \pm 2 \ldots ,$$

the root $\lambda_1 = 0$ being trivial. These are of course the well-known possible forms of the potential near a conducting wedge in a uniform medium. It is also well known that the potential must be finite near the edge, so that the negative roots must be excluded. Thus the root leading to the largest fields near the edge is $\lambda_1 = \frac{\pi}{\alpha}$. The field strength is obtained by differentiation and is hence of the form $r^{(\pi/\alpha)-1}$, which becomes infinite as $r \to 0$, when $\alpha < \pi$.

When $\varepsilon_1 \neq \varepsilon_2$, we must still choose the smallest positive root of equation (2.10). This equation has in fact infinitely many positive roots; their graphical representation is simple. Let $\lambda_1\alpha = x$, so that $\lambda_1(\beta-\gamma) = \left(\frac{\beta-\gamma}{\beta+\gamma}\right) x$. Let $\frac{\beta-\gamma}{\beta+\gamma} = a$, and let $\frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2} = R$. With this notation, equation (2.10) may be written as

(2.11) $$\sin x = R \sin ax .$$

Here the parameters a and R depend only on the ratios β/γ and $\varepsilon_1/\varepsilon_2$ respectively. If we now plot the curves $y = \sin x$, and $y = R \sin ax$ the abcissas x_n of the intersections of these curves provide roots of equation (2.10): we have

(2.12) $$x_1 = \frac{x_n}{\alpha} .$$

Now it is noteworthy that if $R = 0$, or if $a \equiv \frac{\beta-\gamma}{\beta+\gamma} = 0$, then the curve $y = R \sin ax$ degenerates into the axis, so that the roots become

$$\lambda_1 = \frac{n\pi}{\alpha} .$$

The meaning of this is that if the conducting wedge is oriented symmetrically with respect to the interface between the two media, the form of the edge field is the same as in the case where there is no dicontinuity in medium. When neither T nor a is zero, it is important to note that $|R|$ and $|a|$ are less than unity. This means that the first positive root of $y = R \sin ax$ is greater than π. Now we must distinguish two possibilities. First, a and R may be of the same sign. In this event the function $y = R \sin ax$ is positive for $0 < x < \frac{\pi}{|a|}$. This is illustrated in Figure 4.

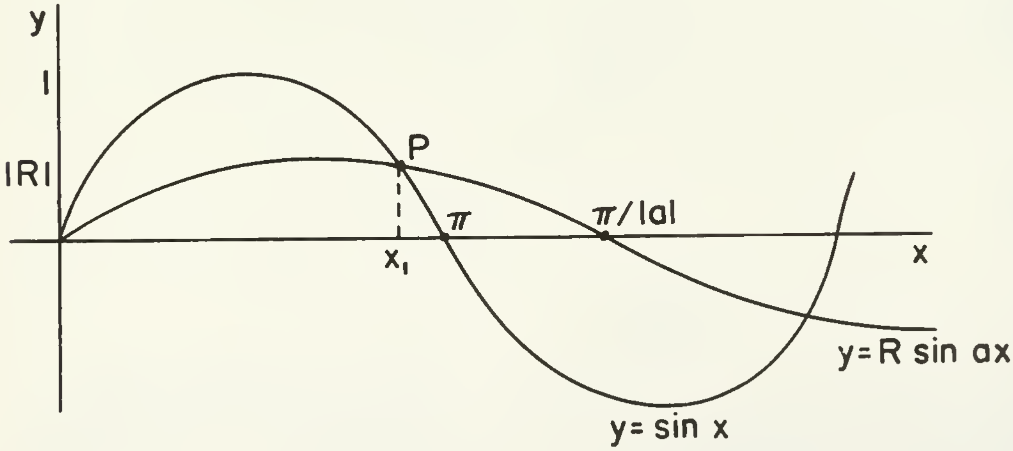

Figure 4

It then appears that $x_1 < \pi$, so that $\lambda_1 < \pi/\alpha$. Since $u = O(r^{\lambda_1})$, $\nabla u = O(r^{\lambda_1 - 1})$, this means that when either $R, a < 0$ or $R, a > 0$ the order of magnitude of the field strength is greater than that of the field strength near the edge of the same conductor when the medium is continuous , and also greater than in the case where the wedge is symmetrically oriented.

Of course if $\alpha > \pi$ then $\nabla u = O(r^{\lambda_1 - 1})$ approaches infinity, and we see that the field strength approaches infinity more rapidly than in the case of a uniform medium, since for the later case $\lambda_1 = \pi/\alpha$. If $\alpha < \pi$ then the field strength in a uniform medium does not tend to infinity, but our field strength still may do so.

Now consider the other possibility, namely that a and R have opposite signs. Then Figure 5 is relevant.

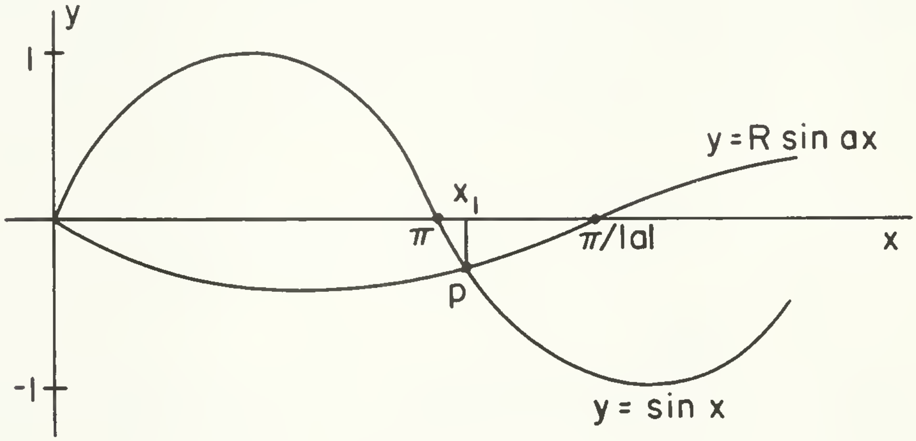

Figure 5

In this case we see that $x_1 > \pi$, and hence $\lambda_1 > \pi/\alpha$, so that the order of magnitude of the edge field is less than the order of magnitude of the edge field in the symmetrical or in the case of a single uniform medium.

The results we have deduced may be characterized very simply if we note that when R and a have the same sign then either $\varepsilon_1 > \varepsilon_2$ and $\beta > \gamma$, or else $\varepsilon_1 > \varepsilon_2$ and $\gamma > \beta$. We therefore have the following simple rule for comparing the singularity in the discontinuous case with the singularity in the continuous case.

Rule: If the medium with greater dielectric constant occupies a greater angular region than the other medium, then the order of singularity at the edge of the conducting wedge is greater then the order of singularity at such an edge immersed in a homogeneous medium. If both media occupy equal angular regions than the singularity is unaltered. If the medium with greater dielectric constant occupies the smaller of the two angular regions, then the singularity is decreased.

[It is interesting to note that we can simulate the regular field behavior in a homogeneous medium near a conducting plane by suitable arrangement of conductor and dielectrics. For example, if we choose $\varepsilon_1/\varepsilon_2 = \sqrt{2+1}/\sqrt{2-1} = 5.83$ and $\alpha = 5\pi/4$, $\beta = 7\pi/8$, $\gamma = 3\pi/8$, then we find $\lambda_1 = 1$.]

The alternatives we have discussed above arise in a natural way. Consider a screen of finite thickness and rectangular cross section, placed, for example, in a cylinder, as illustrated by cross-sectional diagrams of Figures 6a and 6b. Here the hatched portions represent the conducting screen.

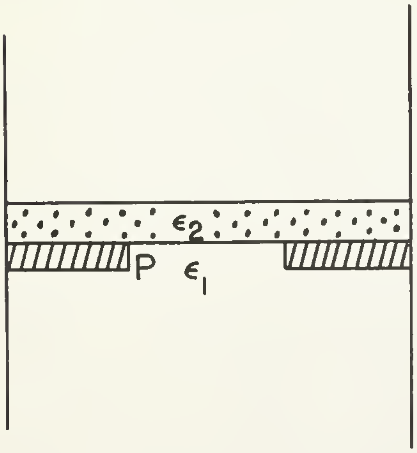

Figure 6a

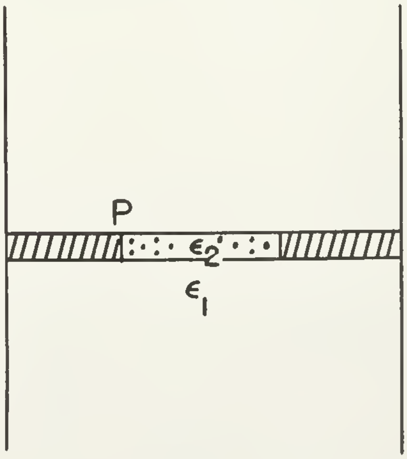

Figure 6b

The lower half of the cylindrical container is filled with a medium of dielectric constant ε_1. The aperture is closed by a slab of dielectric constant ε_2. In Figure 6a we have $\gamma = \pi$, $\beta = \pi/2$, $\alpha = 3\pi/2$. Suppose now $\varepsilon_2 = 1.2$, $\varepsilon_1 = 1$. Call this arrangement a), and let λ_a be corresponding value of λ. Then solving equation (2.11) approximately, we find

$$(2.13) \qquad \lambda_a = .65 < \pi/\alpha = 2/3 = .667 .$$

In Figure 6b we have $\gamma = \pi/2$, $\beta = \pi$, so that

$$(2.14) \qquad \lambda_b = .69 > \pi/\alpha = 2/3 = .667 .$$

Thus the addition of the dielectric slab increases the order of magnitude of the field strength at the relevant corner in arrangement a) while in b) the reverse is the case.

We conclude this section by giving approximate formulas for the calculation of λ when R or a is small. Suppose first that a is small. We consider equation (2.11) and recall that for a = 0, our value of λ is obtained from $x = \pi$, in accordance with equation (2.12). Thus we may now set

$$x = \pi + \delta ; \qquad \lambda = \frac{\pi + \delta}{\alpha} ,$$

where δ is small. Substituting in (2.11) we find

$$\delta = - a \pi R$$

and hence we have

$$(2.15) \qquad \lambda_1 = \frac{\pi}{\alpha} - \frac{a\pi R}{\alpha} \quad \text{for small a} .$$

Since a $\rightarrow$ 0 implies $\beta/\gamma \rightarrow 1$, this equation gives the effect upon λ_1 of a slight departure from a symmetrical orientation of the wedge.

When R is small, i.e. $\varepsilon_1/\varepsilon_2 \rightarrow 1$, we also may write

$$x = \pi + \delta$$

with δ small. We find from (2.11)

$$\sin(\pi+\delta) = R \sin a(\pi+\delta)$$

or

$$-\delta = R\left[\sin \theta\pi + a\delta \cos a\pi\right] .$$

Thus, solving for δ and λ in succession, we find

$$\lambda_1 = \frac{\pi}{a} - \frac{R \sin a\pi}{a} . \tag{2.16}$$

3. Determination of the Green's function

In the present section we determine the Green's function of the configuration of Figure 2. This function is determined by the following equations:

$$\frac{\partial^2 G}{\partial r^2} + \frac{1}{r}\frac{\partial G}{\partial r} + \frac{1}{r^2}\frac{\partial^2 G}{\partial \theta^2} = \frac{\delta(r - r_0)}{r_0}\,\delta(\theta - \theta_0); \qquad \theta \neq \beta , \tag{3.1}$$

$$G(r,0; r_0, \theta_0) = G(r,a; r_0, \theta_0) = 0, \qquad \text{for all } r , \tag{3.2}$$

$$G \rightarrow 0 \qquad \text{as } r \rightarrow \infty , \tag{3.2'}$$

$$\lim_{\delta \rightarrow 0} G(r, \beta-\delta; r_0,\theta_0) = \lim_{\delta \rightarrow 0} G(r,\beta+\delta, r_0,\theta_0) , \tag{3.3}$$

$$\lim_{\delta \rightarrow 0} \varepsilon_1 \frac{\partial G}{\partial \theta}(r,\beta-\delta; r_0,\theta_0) = \lim_{\delta \rightarrow 0} \varepsilon_2 \frac{\partial G}{\partial \theta}(r,\beta+\delta; r_0,\theta_0) . \tag{3.4}$$

Here (r_0,θ_0) are the coordinates of a line source of strength $\frac{1}{2\pi}$ placed in the medium ε_1, and thus we assume

$$(3.5) \qquad 0 < \theta_o < \beta \,.$$

The function G is the potential of the line source, i.e., if r_{PP_o} is the distance between a point $P(r,\theta)$ and a point $P_o(r_o,\theta_c)$ then G behaves like $\frac{1}{2\pi}$ log r_{PP_o} as $P \rightarrow P_o$. The symbol δ denotes the Dirac delta-function.

As a preliminary to the determination of G we define a set of angular eigenfunctions $g_n(\theta)$ by the following equations:

$$(3.6) \qquad g_n''(\theta) + \lambda_n^2 g_n(\theta) = 0 \,; \qquad \theta \neq \beta$$

$$(3.7) \qquad g_n(\theta) = g_n(\alpha) = 0$$

$$(3.8) \qquad \lim_{\delta \to 0} g_n(\beta-\delta) = \lim_{\delta \to 0} g_n(\beta+\delta)$$

$$(3.9) \qquad \lim_{\delta \to 0} \varepsilon_1 \frac{dg_n(\beta-\delta)}{d\theta} = \lim_{\delta \to 0} \varepsilon_2 \frac{dg_n(\beta+\delta)}{d\theta} \,,$$

where the quantities λ_n are to be determined so as to allow the simultaneous fulfillment of equations (3.6) - (3.9).

We proceed somewhat more systematically than in the previous section, and write

$$(3.10) \qquad g_n(\theta) = A_n^{(1)} \sin \lambda_n \theta \,, \qquad 0 \leq \theta < \beta \,,$$

$$(3.10') \qquad g_n(\theta) = A_n^{(2)} \sin \lambda_n\theta + B_n^{(2)} \sin \lambda_n\theta, \qquad \beta < \theta < \alpha \,,$$

thus satisfying equation (2.6) and one of equations (2.7)*. To satisfy the remaining requirements we obtain the homogeneous equations

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* However, cf. equation (3.10'') below, and what follows it.

$$A_n^{(2)} \sin \lambda_n \alpha + B_n^{(2)} \sin \lambda_n \alpha = 0$$

$$A_1^{(1)} \sin \lambda_n \beta - A_n^{(2)} \sin \lambda_n \beta - B_n^{(2)} \sin \lambda_n \beta = 0$$

$$\varepsilon_1 A_1^{(1)} \sin \lambda_n \beta - \varepsilon_2 A_n^{(2)} \sin \lambda_n \beta + \varepsilon_2 B_n^{(2)} \sin \lambda_n \beta = 0 .$$

Setting the determinant of this set of homogeneous equations equal to zero we obtain

$$\begin{vmatrix} 0 & \sin \lambda_n \alpha & \sin \lambda_n \alpha \\ \sin \lambda_n \beta & -\sin \lambda_n \beta & -\cos \lambda_n \beta \\ \varepsilon_1 \cos \lambda_n \beta & -\varepsilon_2 \cos \lambda_n \beta & \varepsilon_2 \sin \lambda_n \beta \end{vmatrix} = 0$$

or

$$(3.11) \qquad \varepsilon_1 \sin \lambda_n \gamma \cos \lambda_n \beta + \varepsilon_2 \cos \lambda_n \gamma \sin \lambda_n \beta = 0 ,$$

where $\gamma = \alpha - \beta$. This is the eigenvalue equation for determination of λ_n, and it agrees with the result of the previous section. Using λ_n so described we can solve for $A_n^{(2)}$, $B_n^{(2)}$ in terms of $A_n^{(1)}$. This is carried out most conveniently by means of the equations

$$A_n^{(2)} \sin \lambda_n \alpha + B_n^{(2)} \cos \lambda_n \alpha = 0$$

$$A_n^{(2)} \sin \lambda_n \beta + B_n^{(2)} \cos \lambda_n \beta = + A_n^{(1)} \sin \lambda_n \beta .$$

We find

$$A_n^{(2)} = \frac{\begin{vmatrix} 0 & \cos \lambda_n \alpha \\ A_n^{(1)} \sin \lambda_n \beta & \sin \lambda_n \beta \end{vmatrix}}{\sin \lambda_n \alpha \cos \lambda_n \beta - \cos \lambda_n \alpha \sin \lambda_n \beta} = - A_n^{(1)} \frac{\cos \lambda_n \alpha \sin \lambda_n \beta}{\sin \lambda_n \gamma} ,$$

$$B_n^{(2)} = \frac{\begin{vmatrix} \sin \lambda_n \alpha & 0 \\ \sin \lambda_n \beta & A_n^{(1)} \sin \lambda_n \beta \end{vmatrix}}{\sin \lambda_n \gamma} = A_n^{(1)} \frac{\sin \lambda_n \alpha \sin \lambda_n \beta}{\sin \lambda_n \gamma} ,$$

and

$$(3.10'') \quad \left.\begin{aligned} g_n(\theta) &= A_n^{(1)} \frac{\sin \lambda_n \beta}{\sin \lambda_n \gamma} \left[- \cos \lambda_n \alpha \sin \lambda_n \theta + \sin \lambda_n \alpha \cos \lambda_n \theta \right] \\ &= A_n^{(1)} \frac{\sin \lambda_n \beta}{\sin \lambda_n \gamma} \sin \lambda_n (\alpha - \theta) \end{aligned} \right\} , \quad \beta \leq \theta \leq \alpha .$$

This result is in natural accord with the definition of $g_n(\theta)$ for $0 < \theta < \beta$, namely*

$$g_n(\theta) = A_n^{(1)} \sin \lambda_n \theta .$$

We now return to make a few remarks about the roots λ_n of the equation (2.11). We have already seen in Section 2 that this equation may be rewritten as

$$(3.12) \qquad \sin \lambda_n \alpha = \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2} \sin \lambda_n (\beta - \gamma) .$$

The first positive root of (2.12) is the quantity λ_1 discussed in Section 2. We observe that the λ_n are connected with the roots of the equation

$$(3.13) \qquad \sin x = R \sin a x ,$$

where

$$R = \frac{(\varepsilon_1/\varepsilon_2) - 1}{(\varepsilon_1/\varepsilon_2) + 1} ,$$

and

$$a = \frac{(\beta/\gamma) - 1}{(\beta/\gamma) + 1} .$$

The roots x_n of this equation (2.13) are the abcissas of the points of intersection of the curves $y = \sin x$ and $y = R \sin a x$. We have $\lambda_n = x_n/\alpha$. If λ is a negative root then $-\lambda$ is a positive root giving the same angular eigenfunction; we can therefore restrict our consideration to the positive roots.

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* Cf. equations (3.10) and (3.10') above.

Let us now consider some integral properties of the functions $g_n(\theta)$. We have in mind the following formulas:

$$\int_0^{\alpha} \varepsilon(\theta) g_m(\theta) g_n(\theta) \, d\theta = 0 , \qquad m \neq n , \tag{3.14}$$

$$\int_0^{\alpha} \varepsilon(\theta) g_n^2(\theta) \, d\theta = A_n^2(1) \left[\frac{\varepsilon_1 \beta \sin^2 \lambda_n \gamma + \varepsilon_2 \gamma \sin^2 \lambda_n \beta}{2 \sin^2 \lambda_n \gamma} \right] . \tag{3.15}$$

Here we have used the notation

$$(3.16) \qquad \begin{aligned} \varepsilon(\theta) &= \varepsilon_1 , \qquad & 0 < \theta < \beta , \\ \varepsilon(\theta) &= \varepsilon_2 , \qquad & \beta < \theta < \alpha . \end{aligned}$$

Equation (3.15) is given for $\sin \lambda_n \gamma \neq 0$. The case $\sin \lambda_n \gamma = 0$ occurs for even n and $\beta = \gamma$ (since then $\lambda_n = n\pi/\alpha$ and $\gamma = \alpha/2$), and will be treated by a limiting operation.

Equations (3.14) and (3.15) may be readily verified by direct integration, using equations (3.10), (3.10'') and (3.11). Incidentally, the subsistence of equation (3.14) suggests itself if one replaces equations (3.6) - (3.9) formally by the equations

$$\frac{d}{d\theta} \left[\varepsilon(\theta) \frac{dg_n}{d\theta} \right] + \lambda_n^2 \varepsilon(\theta) g_n(\theta) = 0, \tag{3.17}$$

$$g_n(0) = g_n(\alpha) = 0 ; \tag{3.18}$$

$$\left[g_n(\beta) \right] = \left[\varepsilon \frac{dg_n}{d\theta} \right]_{\theta=\beta} = 0 , \tag{3.18'}$$

where the symbol [ ] stands for the discontinuity of the function included in the brackets.

We now define a suitable finite Fourier transformation by the equation

$$(3.19)\qquad \int_0^{\alpha} \varepsilon(\theta) G(r,\theta;\, r_o,\theta_o)\, g_n(\theta)\, d\theta \;=\; v_n(r; r_o,\theta_o) \;.$$

(For brevity, we shall usually write $v_n(r)$, or merely v_n, instead of $v_n(r;\, r_o,\theta_o)$.)

Multiply equation (3.1) by $\varepsilon(\theta) g_n(\theta)$, and integrate first over the interval $0 < \theta < \beta-\delta$, and then over the interval $\beta+\delta < \theta < \alpha$. We find:

$$(3.20)\quad \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right) \int_0^{\beta-\delta} G(r,\theta)\, \varepsilon_1 g_n(\theta)\, d\theta + \frac{1}{r^2}\int_0^{\beta-\delta} \frac{\partial^2 G}{\partial\theta^2}\, \varepsilon_1 g_n(\theta)\, d\theta = \varepsilon_1 \frac{\delta(r-r_o)}{r_o}\, g_n(\theta_o),$$

$$(3.21)\quad \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right) \int_{\beta+\delta}^{\pi} G(r,\theta)\, \varepsilon_2 g_n(\theta)\, d\theta + \frac{1}{r^2}\int_{\beta+\delta}^{\pi} \frac{\partial^2 G}{\partial\theta^2}\, \varepsilon_1 g_n(\theta)\, d\theta \;=\; 0 \;.$$

Now we integrate by parts, to obtain

$$(3.22)\quad \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right) \int_0^{\beta-\delta} \varepsilon_1 G g_n\, d\theta + \frac{\varepsilon_1}{r^2}\left[\left(g_n \frac{\partial G}{\partial\theta} - G g_n'\right)\Bigg|_0^{\beta-\delta} + \int_0^{\beta-\delta} G\, g_n''\, d\theta\right] = \varepsilon_1 \frac{\delta(r-r_o)}{r_o}\, g_n(\theta_o),$$

$$(3.23)\quad \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} \int_{\beta+\delta}^{\alpha} \varepsilon_2 G\, g_n\, d\theta + \frac{\varepsilon_2}{r^2}\left[\; g_n \frac{\partial}{\partial\theta} - G g_n' \;\Bigg|_{\beta+\delta}^{\alpha} + \int_{\beta+\delta}^{\alpha} G\, g_n''\, d\theta \right] \;=\; 0$$

These transformations of (3.20) and (3.21) are permissible since $\theta = \beta$ in the present ranges. Now we transform the last term in Equations (3.22) and (3.23) by use of (3.6), and add the resulting equations. We use the boundary conditions on g_n and G at $\theta = 0$, $\theta = \alpha$, and find

(3.24)
$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{\lambda_n^2}{r^2}\right)\left[\int_0^{\beta-\delta}\varepsilon(\theta)G\ g_n d\theta+\int_{\beta+\delta}^{\alpha}\varepsilon(\theta)G\ g_n d\theta\right]$$
$$+\frac{1}{r^2}\left\{\varepsilon_1 g_n(\beta-\delta)\frac{\partial G}{\partial\theta}(\theta,\beta-\delta)-\varepsilon_1 G(\theta,\beta-\delta)g_n'(\beta-\delta)\right.$$
$$\left.-\varepsilon_2 g_n(\beta+\delta)\frac{\partial G}{\partial\theta}(\theta,\beta+\delta)+\varepsilon_2 G(r,\beta+\delta)g_n'(\beta+\delta)\right\}$$
$$=\varepsilon_1\frac{\delta(r-r_o)g_n'(\theta_o)}{r_o}$$

(for brevity we have omitted mention of r_o, θ_o in listing the arguments of G, $\frac{\partial G}{\partial\theta}$). This equation holds for every δ, however small. We may therefore pass to the limit $\delta \to 0$. Using equations (3.3), (3.4), (3.8) and (3.9), we find that the espression in curly brackets in (3.24) approaches zero; the expression in square brackets becomes $v_n(r)$ as defined in (3.19). Thus we have

(3.25)
$$\frac{d^2v_n}{dn^2}+\frac{1}{r}\frac{dv_n}{dn}-\frac{\lambda_n^2 v_n}{r^2} = \varepsilon_1\frac{\delta(r-r_o)}{r_o}\ g_n(\theta_o), \qquad 0<\theta_o<\beta .$$

The delta-function inhomogeneity in equation (3.25) implies that v_n is a solution of the corresponding homogeneous equation for $r \neq r_o$, and also implies the following limiting conditions for $r \to r_o$:

(3.26)
$$\lim_{\delta\to 0}\left[v_n(r_o+\delta)-v_n(r_o-\delta)\right] = 0,$$

(3.27)
$$\lim_{\delta\to 0}\left[v_n'(r_o+\delta)-v_n'(r_o-\delta)\right] = \frac{\varepsilon_1 g_n(\theta_o)}{r_o} .$$

We require $v_n(0)$ and $v_n(\infty)$ to be finite in view of (3.19), (3.2) and (3.2'). Hence we choose $v_n(r) = A_n r^{\lambda_n}$ for $r < r_o$, and $v_n = B_n r^{-\lambda_n}$ for $r > r_o$, since these functions are the finite solutions of (3.25) for $r \neq r_o$. To satisfy (3.26) we must have $A_n r_c^{\lambda_n} = B_n r_o^{-\lambda_n} = C_n$, say. Then we have

(3.28)
$$v_n(r;\, r_o) = C_n \left(\frac{r}{r_o}\right)^{\lambda_n} \qquad \text{for } r < r_o ,$$
$$v_n(r;\, r_o) = C_n \left(\frac{r_o}{r}\right)^{\lambda_n} \qquad \text{for } r > r_o .$$

We now determine C_n by means of equation (3.27). This gives

$$\text{(3.29)} \qquad \lim_{\delta \to 0} C_n \left[r_o^{\lambda_n} \cdot (-\lambda_n) \cdot (r_o + \delta)^{-\lambda_n - 1} - r_o^{-\lambda_n} \cdot (+\lambda_n) \cdot (r_o - \delta)^{\lambda_n - 1} \right] = \frac{\varepsilon_1 g_n(\theta_o)}{r_o} ,$$

or

$$\text{(3.30)} \qquad C_n = -\frac{\varepsilon_1 g_n(\theta_o)}{2\lambda_n} .$$

Combining equations (3.19), (3.28) and (3.30), we find

(3.31)
$$\int_0^{\alpha} \varepsilon(\theta) G(r,\theta;r_o,\theta_o) g_n(\theta)\, d\theta = v_n(r;r_o,\theta_o) = -\frac{\varepsilon_1 g_n(\theta_o)}{2\lambda_n} \left(\frac{r}{r_o}\right)^{\lambda_n}, \qquad \text{for } r < r_o ,$$
$$= \frac{-\varepsilon_1}{2\lambda_n} g_n(\theta_o) \left(\frac{r_o}{r}\right)^{\lambda_n}, \qquad \text{for } r > r_o .$$

Now consider the function

$$H = \sum_1^{\infty} \frac{v_m(r,r_o)}{\Gamma_m} g_m(\theta) ,$$

where $\Gamma_m = \int_0^{\alpha} \varepsilon(\theta) g_m^2(\theta)\, d\theta$ is a quantity evaluated in equation (3.15). If we multiply H by $g_n(\theta)\,\varepsilon(\theta)$ and integrate from zero to α, we find, by applying the orthogonality relation (3.14), that

$$\int_0^{\alpha} \varepsilon(\theta)\, H\, g_n(\theta)\, d\theta = v_n(r) .$$

We assume that the $g_n(\theta)$ form a complete set, and then it follows that $G = H$, i.e.,

$$(3.32) \qquad G(r,r_o;\,\theta,\theta_o) = -\sum_{r=1}^{\infty} \frac{v_n(r,r_o)}{\Gamma_n}\, g_n(\theta,\theta_o) \,,$$

or

$$(3.33) \quad G(r,r_o;\theta,\theta_o) = -\,\varepsilon_1 \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \left[\frac{\sin^2 \lambda_n \gamma}{\varepsilon_1 \beta \sin^2 \lambda_n \gamma + \varepsilon_2 \gamma \sin^2 \lambda_n \beta} \right] \left(\frac{r}{r_o}\right)^{\pm\lambda_n} \frac{g_n(\theta) g_n(\theta_o)}{\left[A_n^{(1)}\right]^2} \,,$$

$0 < \theta_o < \beta$, where in the exponent of $\frac{r}{r_o}$, the positive sign is to be employed when $r < r_o$ and the negative sign when $r > r_o$. This result has been derived for the case where θ_o, the angular coordinate of the line source, is in the ε_1 medium. For the definitions of $g_n(\theta)$ and of the quanities λ_n see (3.10), (3.10') and (3.11). It should be noted that since $g_n(\theta)$ and $g_n(\theta_o)$ in (3.33) each contain $A_n^{(1)}$ as a factor, equation (3.33) is independent of $A_n^{(1)}$. In medium I, we have $g_n(\theta)\, g_n(\theta_o) \big/ \left[A_n^{(1)}\right]^2 = \sin \lambda_n \theta \sin \lambda_n \theta_o$. It should be noted here that an analysis similar to that above can be given* for conical geometries, as well as for the case where

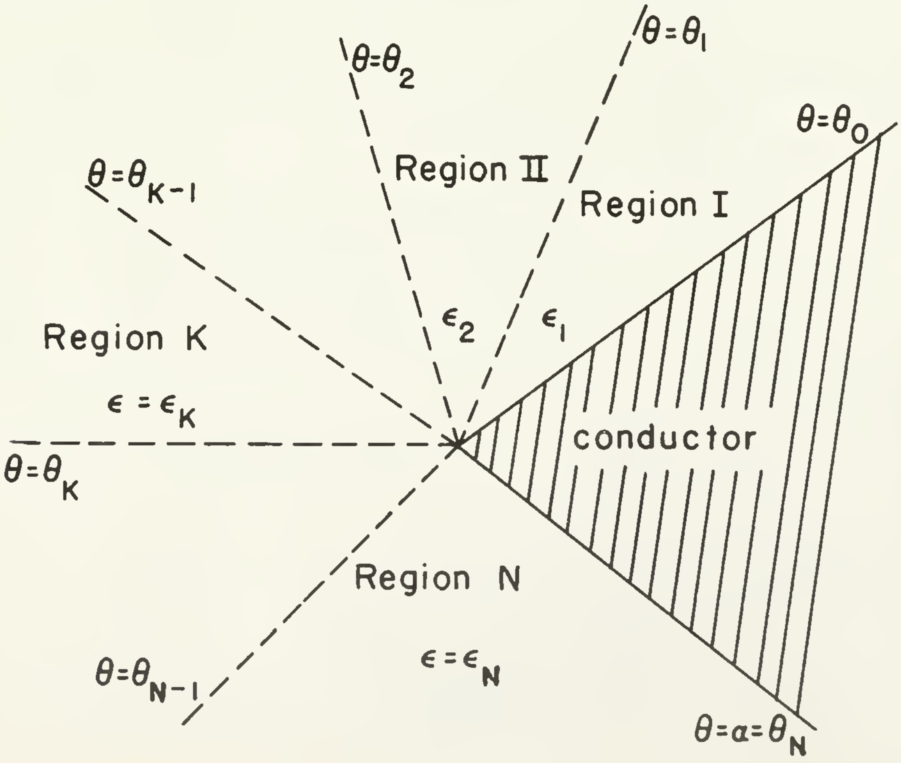

Figure 7

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* In concial geometries the Legendre functions $P_{\lambda n}$ (cos θ) replace the trigonometric functions we have employed, and the λ_n are determined again by the boundary and interface conditions.

there are n different dielectrics present. In figure 7 we illustrate the case where the exterior of conducting wedge is filled by n wedges of different dielectric constants. Here the shaded wedge represents a conductor, and the regions I,II,K,N, contain media of dielectric constants ε_1, $\varepsilon_2 \ldots$, $\varepsilon_K \ldots$, ε_N, respectively. Define $\varepsilon(\theta) = \varepsilon_K$ for $\theta_{K-1} < \theta < \theta_K$, for $1 \leq K < N$. Functions $g_n(\theta)$ may be defined by the equation $g_n'' + \lambda_n^2 g_n = 0$ for $\theta \neq \theta_1, \theta_2, \ldots, \theta_N$, together with the equations $g_n(\theta) = g_n(\alpha) = 0$ and the conditions on the continuity of g_n and of $\varepsilon(\theta) dg_n/d\theta$. In the K^{th} region we can set

$$g_n(\theta) = A_n^{(K)} \sin \lambda_n \theta + B_n^{(K)} \cos \lambda_n \theta , \qquad \theta_{K-1} < \theta < \theta_K .$$

We will have N-1 homogeneous equations expressing continuity of $g_n(\theta)$ at $\theta = \theta_1, \ldots, \theta_{N-1}$, and N-1 homogeneous equations expressing continuity of $\varepsilon(\theta) dg_n/d\theta$. Together with the conditions $g_n(\theta) = 0$, $g_n(\alpha) = 0$, we will then have 2N homogeneous equations for the 2N unknowns $A_n^{(K)}$, $B_n^{(K)}$, for $1 \leq K \leq N$. The λ_n must be chosen as the roots of the determinantal equation expressing the fact that these equations have a non-trivial solution. But it is not our purpose to pursue this matter further here.

We direct attention now to equation (3.32) and calculate the limit of G as $\beta \rightarrow \gamma$ (cf. Fig. 3), i.e., $\beta \rightarrow \alpha/2$, $\gamma \rightarrow \alpha/2$. To do this we let $\gamma \rightarrow \beta$, while $\gamma + \beta$ is constantly equal to α. Then we note first that $\lambda_n \rightarrow \frac{n\pi}{\alpha}$, since the equation for λ_n is

$$\varepsilon_1 \sin \lambda_n \gamma \cos \lambda_n \beta + \varepsilon_2 \sin \lambda_n \beta \cos \lambda_n \gamma = 0 ,$$

which becomes, in the limit,

$$\left(\frac{\varepsilon_1 + \varepsilon_2}{2} \right) \sin 2\lambda_n \gamma = 0 , \tag{3.11}$$

so that

$$2\lambda_n \gamma = n\pi,$$

$$\lambda_n = \frac{n\pi}{\alpha} .$$

Now we must consider the limiting form of the coefficient

$$(3.34)\qquad \frac{-\varepsilon_1 \sin^2 \lambda_n\gamma}{(\lambda_n)\left[\varepsilon_1\beta \sin^2\lambda_n\gamma + \varepsilon_2\gamma \sin^2\lambda_n\beta\right]} \equiv E_n \, .$$

When $\gamma \rightarrow a/2$ and $\beta \rightarrow a/2$ and n is odd, then $\lambda_n\gamma \rightarrow \frac{n\pi}{2}$, $\lambda_n\beta = \frac{n\pi}{2}$, and there is no difficulty in taking the limit. We obtain

$$(3.35)\qquad E_n \rightarrow \frac{-\varepsilon_1}{n\pi/a} \frac{1}{(\varepsilon_1 + \varepsilon_2)a/2} = -\frac{1}{\pi}\frac{2\varepsilon_1}{\varepsilon_1 + \varepsilon_2} \cdot \frac{1}{n}, \qquad n \text{ odd} \, .$$

But when n is even, the numerator and denominator both tend to vanish, (i.e., if $n = 2p$, with p integral, then $\sin \lambda_n\beta \rightarrow \sin 2\pi p/2a = 0$). In order to obtain the correct limit we note that by the eigenvalue equation (3.11) we have

$$(3.36)\qquad \frac{\sin^2 \lambda_n\beta}{\sin^2 \lambda_n\gamma} = \frac{\varepsilon_1^2 \cos^2 \lambda_n\beta}{\varepsilon_2^2 \cos^2 \lambda_n\gamma} \, .$$

Now when $\sin \lambda_n\beta$, $\sin \lambda_n\gamma \rightarrow 0$, then the quotient of the corresponding squared cosines tends to unity if n is even since $\lambda_n\beta$ and $\lambda_n\gamma$ tend to integral multiples of π. Thus

$$\lim_{\beta \rightarrow \gamma} \left(\frac{\sin^2 \lambda_n\beta}{\sin^2 \lambda_n\gamma}\right) \rightarrow \frac{\varepsilon_1^2}{\varepsilon_2^2} \, .$$

Now dividing the numerator and denominator of (3.34) by $\sin^2 \lambda_n\gamma$, and using the correct limiting form of $\sin^2 \lambda_n\beta/\sin^2 \lambda_n\gamma$, we find

$$(3.37)\qquad \lim_{\beta \rightarrow \gamma} E_n = -\frac{1}{\pi}\frac{2\varepsilon_2}{\varepsilon_1 + \varepsilon_2} \cdot \frac{1}{n}, \qquad \text{for } n \text{ even} \, .$$

Thus, for $\beta = \gamma$, we have the following formula for G:

$$G = -\frac{1}{\pi} \sum_{p=1}^{\infty} \frac{2\varepsilon_2}{\varepsilon_1 + \varepsilon_2} \frac{1}{2p-1} \left(\frac{r_0}{r}\right)^{\pm(2p-1)\pi/\alpha} \cdot \sin(2p-1)\frac{\pi\theta}{\alpha} \sin(2p-1)\frac{\pi\theta_0}{\alpha}$$

$$-\frac{1}{\pi} \sum_{p=1}^{\infty} \frac{2\varepsilon_2}{\varepsilon_1 + \varepsilon_2} \frac{1}{2p} \left(\frac{r_0}{r}\right)^{\pm 2p\pi/\alpha} . \sin \frac{2p\pi}{\alpha} \theta \sin \frac{2p\pi}{\alpha} \theta_0 ,$$

for θ, θ_0 in Region I.

On the other hand, let us consider the Green's function for the wedge when $\varepsilon_1 = \varepsilon_2$. Calling this G_0, we have from the above equation

$$G_0 = -\frac{1}{\pi} \sum_{p=1}^{\infty} \frac{1}{(2p-1)} \left(\frac{r_0}{r}\right)^{\pm(2p-1)\pi/\alpha} \cdot \sin(2p-1) \frac{\pi}{\alpha} \theta \sin \frac{2p-1}{\alpha} \pi \theta_0$$

$$-\frac{1}{\pi} \sum_{p=1}^{\infty} \frac{1}{2p} \left(\frac{r_0}{r}\right)^{2p\pi/\alpha} \cdot \sin \frac{2p\pi}{\alpha} \theta \sin \frac{2p\pi}{\alpha} \theta .$$

If we write $G = G_0 + \delta G$, where δG is the change in the Green's function produced by the change in dielectric, we have

$$\delta G \equiv G - G_0 = -\frac{1}{\pi} \sum_{p=1}^{\infty} \left(\frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2}\right) \frac{1}{2p-1} \left(\frac{r}{r_0}\right)^{\pm(2p-1)\pi/\alpha} \cdot \sin(2p-1)\frac{\pi\theta}{\alpha} \sin(2p-1) \frac{\pi\theta_0}{\alpha}$$

(3.39)

$$-\frac{1}{\pi} \sum_{p=1}^{\infty} -\left(\frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2}\right) \frac{1}{2p} \left(\frac{r}{r_0}\right)^{2p\pi/\alpha} \cdot \sin 2p \frac{\pi\theta}{\alpha} \sin 2p \frac{\pi\theta_0}{\alpha} ,$$

or

$$\delta G = -\frac{1}{\pi} \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2} \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{r}{r_0}\right)^{\pm n\pi/\alpha} (-1)^{n+1} \circ \sin n \frac{\pi\theta}{\alpha} \sin n \frac{\pi\theta_0}{\alpha} .$$

Noting that

$$(-1)^{n+1} \sin \frac{n\pi}{\alpha} \theta_0 = \sin \frac{n\pi}{\alpha} (\alpha - \theta_0) ,$$

we have

(3.40) $$\delta G = \left(\frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2}\right)\left(-\frac{1}{\pi}\right)\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{r_o}{r}\right)^{\pm n\pi/\alpha}\cdot\sin\frac{n\pi}{\alpha}\theta\,\sin\frac{n\pi}{\alpha}(\alpha-\theta_o)\,.$$

If we exhibit the arguments of G_o explicitly by using the notation $G_o \equiv G_o(r,\theta;\, r_o,\theta_o)$, we can write

(3.41) $$\delta G = \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2}\, G_o(r,\theta;\, r_o,\alpha-\theta_o)\,,$$

so that we have

(3.42) $$G = G_o(r,\theta;\, r_o,\theta_o) + \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2}\, G_o(r,\theta;\, r_o,\alpha-\theta_o)\,,$$

for θ, θ_o in Region I.

It is not difficult to show that in Region II we have, in addition,

(3.42') $$G = G_o + \frac{2\varepsilon_1}{\varepsilon_1 + \varepsilon_2}\, G_o(r,\theta;\, r_o,\, \alpha-\theta_o)\,.$$

Now the point $(r_o,\alpha-\theta_o)$ is the reflection of the point r_o,θ_o in the horizontal line $\theta = \alpha/2$, so that the solution G has been expressed here, as it were, by means of images. This is the result we were working towards in the immediately preceding calculation. It constitutes a generalization of a well-known result of Kelvin. Kelvin observed that the field of a point charge placed over a plane boundary separating two homogeneous media with different dielectric constants can be expressed by means of images. Let ε_1 be the dielectric constant of the medium or half-space in which the source is placed, and let ε_2 be the dielectric constant of the other half-space; further, let G(P,Q) be the three-dimensional field at P of the point source placed at point Q, and let Q' be the image of Q in the interface, and let $R = \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2}$, $T = \frac{2\varepsilon_1}{\varepsilon_1 + \varepsilon_2}$. The medium of dielectric constant ε_1 shall be denoted by I and the medium of dielectric constant ε_2 by II, and R(P,Q) shall be the distance between the points P and Q. Then, according to Kelvin

$$G(P,Q) = \frac{1}{R(P,Q)} + R \cdot \frac{1}{R(P,Q')} \qquad \text{in I}$$

$$= \frac{T}{R(P,Q)} \qquad \text{in II} .$$

Now $\frac{1}{R_{PQ}}$ is the Green's function for free space in three dimensions, but Kelvin's result obviously holds also for two dimensions, if we replace $1/R(p,Q)$ by the two-dimensional free-space Green's function. It is now clear that our result is a simple generalization of the two-dimensional form of Kelvin's theorem to our configuration. In the generalization, the coefficients R and T are preserved, and the free-space Green's function is merely replaced by the Green's function for the wedge.

We have found that by extending the meaning of G_0 the present formulas (3.42) and (3.42') can be still further generalized to apply to three dimensions. Furthermore the configuration of conductors is only restricted by the requirement of symmetry with respect to the interface. In fact the theorem is actually a consequence of symmetry considerations alone, as will be shown in the next section.

4. Generalization of Kelvin's theorem

In the present section we prove a simple generalization of the final result of the previous calculations.

We consider a distribution of conductors symmetrically placed with respect to a horizontal plane, say the plane $z = 0$. Such a situation is schematically depicted in Figure 8.

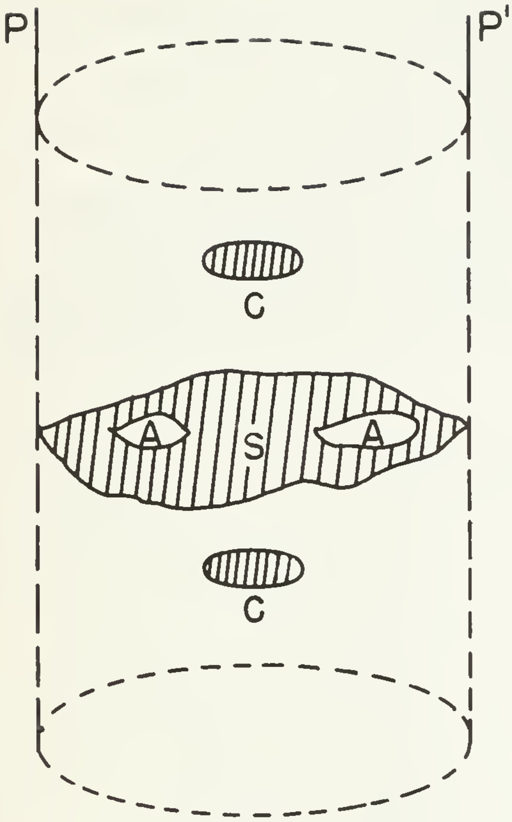

Figure 8

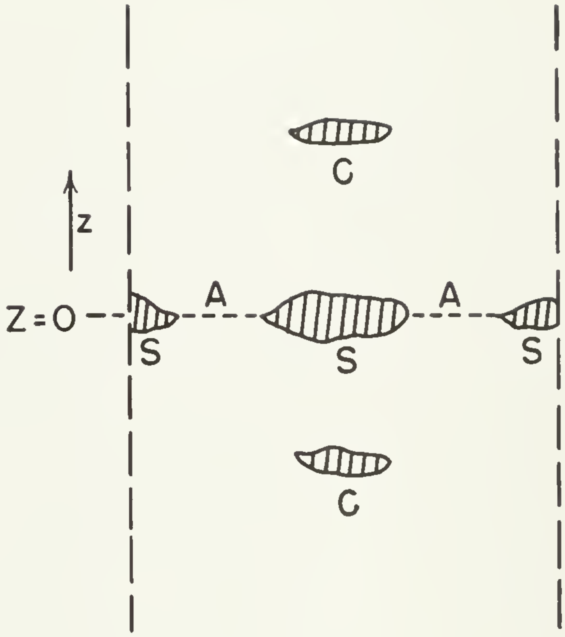

Figure 8a

Here the parts marked C are conductors, and they include, for the sake of completeness, a cylindrical boundary. In addition to the conductors C we single out for special attention a conductor S of arbitrary plan form. If S is not of zero thickness then it is required to be symmetric with respect to the plane z = 0. The surface S will be called the 'screen' and it is assumed to possess apertures which are denoted by the letter A in the figure. Except for the requirement of symmetry with respect to the plane z = 0 the conductor may possess any shape whatsoever. We require, however, that the electrostatic Green's functions for the configuration C and for the total configuration C + S exist uniquely for the case when the configurations in question are immersed in a homogeneous medium of dielectric constant ε_1. These Green's functions we denote by $\gamma(P,Q)$ and $g(p,Q)$ respectively, and they represent the electrostatic potential at a point P due to a point charge at the point Q in the presence of C or C + S respectively. (It may be remarked that such functions are explicitly known only for relatively few configurations. Experience with these known cases shows that as the point P approaches sharp edges or spikes of the conductors, the electric field tends to infinity in a manner which depends strongly on the shape of the edge or spike.)

In addition to the situation described above we consider an altered situation in which the dielectric constant is still ε_1 for $z > 0$, but changes abruptly to the value ε_2 for $z < 0$. Then the new Green's function for the configuration C is denoted by $\Gamma(P,Q)$ and the new Green's function for the configuration C + S is denoted by $G(P,Q)$. Let the region $z > 0$ be called Region I and the region $z < 0$ be called Region II. Now let Q' be the reflection of the point Q in the plane $z = 0$. We wish to prove the following theorem:

Theorem:

$$\begin{aligned} G(P,Q) &= g(P,Q) + Rg(P,Q') \qquad && z > 0 \\ G(P,Q) &= Tg(P,Q) && z < 0 \,. \end{aligned} \tag{4.1}$$

Here R and T are the same constants that occur in Kelvin's theorem, namely

$$R = \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2} \;; \qquad T = \frac{2\varepsilon_1}{\varepsilon_1 + \varepsilon_2} \,. \tag{4.2}$$

The proof is easy, and does not depend upon an explicit knowledge of $g(P,Q)$. Consider Kelvin's theorem, which states that the field of a point charge in the presence of a distribution of dielectric constant $\varepsilon(z)$ [where $\varepsilon(z) = \varepsilon_1$ for $z > 0$ and $\varepsilon(z) = \varepsilon_2$ for $z < 0$] is given by the formula

$$(4.3) \qquad \begin{aligned} G &= \frac{1}{R(P,Q)} + \frac{1}{R(P,Q')} \,, && z > 0 \,, \\ G &= \frac{T}{R(P,Q)} && z < 0 \,. \end{aligned}$$

Here $R(P,Q)$ means the distance from the field point P to the point Q at which the point charge is located. R and T are constants. The function described in (4.1) and (4.2), cleraly obeys Laplace's equation in Regions I and II. It has the characteristic singularity at the point Q, and is regular elsewhere and vanishes suitably at infinity. Thus one has only to verify the conditions that G and $\varepsilon \frac{\partial G}{\partial z}$ be continuous at the plane $z = 0$.

The correctness of Kelvin's theorem, (4.3), is then seen to depend on the easily verified fact that at $z = 0$, we have

$$(4.3)' \qquad \frac{1}{R(P,Q)} = \frac{1}{R(P,Q')} \;:\; \frac{\partial}{\partial z}\left[\frac{1}{R(P,Q)}\right] = -\frac{\partial}{\partial z}\left[\frac{1}{R(P,Q')}\right] .$$

In fact the interface conditions for $G(P,Q)$ are

$$(4.3)''. \quad \lim_{z \to 0+} G(P,Q) = \lim_{z \to 0-} G(P,Q) : \quad \lim_{z \to 0+} \varepsilon_1 \frac{\partial G}{\partial z}(P,Q) = \lim_{z \to z-} \varepsilon_2 \frac{\partial G}{\partial z}(P,Q) .$$

If we now insert the formulas (4.3) for G into equation $(4.3)''$ we find

$$\left.\begin{aligned} &(4.4) && \frac{1}{R(P,Q)} + \frac{R}{R(P,Q')} = \frac{T}{R(P,Q)} . \\ &(4.5) && \varepsilon_1 \frac{\partial}{\partial z}\frac{1}{R(P,Q)} + \varepsilon_1 \frac{R\partial}{\partial z}\frac{1}{R(P,Q')} = \varepsilon_2 \frac{T\partial}{\partial z}\frac{1}{R(P,Q)} \end{aligned}\right\} z = 0 .$$

Then the correctness of equations $(4.3)'$ ensures that a common factor can be removed from each of the equations(4.4) and (4.5), so that we have a system of linear equations for the constants R and T.

We find

$$(4.6)\qquad \begin{aligned} 1 + R &= T , \\ \varepsilon_1 (1 - R) &= \varepsilon_2 T \end{aligned}$$

with the solution

$$(4.7)\qquad \begin{aligned} R &= \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2} \\ T &= \frac{2\varepsilon_1}{\varepsilon_1 + \varepsilon_2} \quad . \end{aligned}$$

Now the proof of our theorem is completely analogous to the proof of Kelvin's theorem. In fact, from considerations of symmetry we note that

$$(4.8)\qquad g(x_p,y_p,z_p; x_Q,y_Q,z_Q) = g(x_p,y_p,-z_p; x_Q,y_Q,-z_Q) \quad .$$

[In this equation we have exhibited the arguments of g in detail. The subscripts p and Q designate the coordinates of the points p and Q respectively; the coordinates of the point Q' are $(x_Q,y_Q,-z_Q)$.] Now, in the apertures A, we have, by the continuity of g, and by (4.8)

$$(4.9)\qquad g(x_p,y_p,0; x_Q,y_Q,z_Q) = g(x_p,y_p,0; x_Q,y_Q,-z_Q) \quad .$$

Also, forming the z-derivatives, and using (4.8) we find in the apertures

$$(4.10)\qquad \frac{\partial g}{\partial z} (x_p,y_p,0; x_Q,y_Q,z_Q) = - \frac{\partial g}{\partial z} (x_p,y_p,0; x_Q,y_Q,-z_Q) \quad .$$

(here we have employed Eq. (4.8)). Hence the function g has those properties of the function $\frac{1}{R(P,Q)}$ which were essential in treating continuity at the interface for Kelvin's theorem, so that our functions G and $\varepsilon \frac{\partial G}{\partial z}$ (as defined in equations (4.1) and (4.2)) are continuous in the aperture. That G vanishes on all conductors as required is ensured by the definition of g(P,Q) and $g(P,Q')$

The conditions that G must have a singularity at $P = Q$ and vanish at infinity (if infinity is accessible in the configuration in question) are similarly satisfied. In addition the behavior of G at geometric singularities of the conductors is the same as the behavior of the function g. Since g is assumed to be satisfactory in this regard so is G. Hence, by the postulated uniqueness of the function G defined in (4.1) and (4.2), that the function is the Green's function of our problem and our theorem is verifed.

5. Behavior of the singularity at the interface

From the foregoing theorem we may now draw some conclusions as to behavior near singular points of conductors.

Let us consider the case where in Figs. 8a and 8b, S is a *plane* screen of zero thickness and arbitrary shape. We can write

$$\begin{aligned} g(P,Q) &= \gamma(P,Q) + h(P,Q) , \\ g(P,Q') &= \gamma(P,Q') + h(P,Q'), \end{aligned} \tag{5.1}$$

where $\gamma(P,Q)$ is the Green's function for the configuration C in Figs. 8a and 8b. Equations (1) defines the function $h(P,\Theta)$ as a regular function. Now when P is a point on the screen, then $g(P,Q)$ and $G(P,Q')$ both vanish, and hence $h(P,Q) = -\gamma(P,Q)$ and $h(P,Q') = -\gamma(P,Q')$. But if P is on S, it is in the plane $z = 0$, and it is easy to see that for such a point we have $\gamma(P,Q) = \gamma(P,Q')$. It follows that $h(P,Q) = h(P,Q')$ when the point P is on S. Now $g(P,Q)$ and $\gamma(P,Q)$, and also $g(P,Q')$ and $\gamma(P,Q')$ must vanish on the remaining conductors C, and hence so must $h(P,Q)$ and $h(P,Q')$. If infinity is accessible, $h(P,Q)$ and $h(P,Q')$ must vanish there since both $g(P,Q)$ and $\gamma(P,Q)$ vanish. Hence $h(P,Q) - h(P,Q')$ is a function of P which vanishes on all conductors in the configuration and at infinity (if infinity is accessible) and is regular elsewhere. If such a function were not identically zero it coud be added to $g(P,Q)$; but this would contradict the postulated uniqueness of the latter. Therefore $h(P,Q) - h(P,Q') \equiv 0$, or

$$h(P,Q) \equiv h(P,Q') \,. \tag{5.2}$$

Now let us define $H(P,Q)$ by the equation

$$G(P,Q) = \Gamma(P,Q) + H(P,Q) \,, \tag{5.3}$$

where $\Gamma(P,Q)$ and $G(P,Q)$ have been previously defined as the Green's function for C and C + S respectively, when ε is discontinuous. We can easily show that

$$\begin{aligned} \Gamma(P,Q) &= \gamma(P,Q) + R\gamma(P,Q') \,, && z > 0 \\ &= T\gamma(P,Q) \,, && z < 0 \,. \end{aligned} \tag{5.4}$$

The proof is the same as in the case of the theorem of Equation (4.1).

Now using (5.1) and (5.2) we rewrite equation (4.1) as follows:

$$\begin{aligned} G(P,Q) &= \gamma(P,Q) + h(P,Q) + R\left[\gamma(P,Q') + h(P,Q)\right] && z > 0 \\ &= T\left[\gamma(P,Q) + h(P,Q)\right] \,, && z < 0 \end{aligned} \tag{5.5}$$

Comparing with (5.4) we find

$$H(P,Q) = (1 + R)\, h(P,Q) = Th(P,Q) = \frac{2\varepsilon_1}{\varepsilon_1 + \varepsilon_2}\, h(P,Q) \,. \tag{5.6}$$

But since $\Gamma(P,Q)$ refers to the configuration C alone, any infinities near S in the field arising from the potential C are only due to the partial potential $H(P,Q)$. The same remark holds for the fields arising from the potentials $\gamma = g+h$.

Hence if any infinities actually occur in the field strengths as P approaches a point of S, then in such an approach ∇H and ∇h are the dominant parts of ∇G and ∇g.

We infer that if in the case of a homogeneous medium the field strength tends to infinity as a certain point of the conducting screen is approached, then the same is true when the medium is the discontinuous one considered in this investigation. Furthermore, the manner in which the field strength tends to infinity is the same in both cases, and we have

$$\lim_{p \to S} \frac{\nabla G}{\nabla g} \to \frac{2\varepsilon_1}{\varepsilon_1 + \varepsilon_2} \quad . \tag{5.7}$$

[Here the meaning of the vector quotient is that the ratios of corresponding components are to be taken.]

Since no general classification of singular shapes of conductors and of the corresponding local functional forms of the fields is known, the present result would seem to be of special interest. It tells us the effect of introducing a discontinuity of dielectric constant even when the original local form of the field is not known analytically. The above results referred to a screen of zero thickness, and arbitrary shapes. In the case of a symmetrical screen of non-zero thickness, equation (5.2) is no longer valid. One can still show, however, that

$$H(P,Q) = h(P,Q) + Rh(P,Q') = h(P,Q) + Rh(P',Q), \qquad z > 0 , \tag{5.8}$$

$$H(P,Q) = T\,h\,(P,Q) , \qquad z < 0 . \tag{5.9}$$

Thus Equation (5.7) may be deduced as before for $z < 0$, i,e., in Region II, (the source being in Region I). The same conclusion as before also holds for approach to S along the interface. For z positive we believe the result (5.7) may still hold, although this has not yet been proven for general S.

6. Conclusions

We have determined the behavior of an electrostatic field near the vertex of a wedge-shaped conductor in the presence of a medium whose dielectric constant is a step function of angle as measured with respect to a pole at the vertex of the wedge. The field becomes singular as the edge is approached, and is of the form of $r^{\lambda-1}$, where r is the distance from the vertex and where λ depends upon the values of the dielectric constant and also upon the angle. Also the Green's function has been derived for the case when the interface is inclined at an arbitrary angle. Its form provides an expansion of the solution near the edge in a certain set of powers of the distance. Accordingly as the medium of greater dielectric constant occupies the greater (or smaller) angle, the singularity at the edge is greater (or smaller) than it would be in the case of a homogeneous medium. For the symmetrical case, i.e., when the interface bisects the exterior angle of the wedge, the form of the singularity is identical with the form obtained in the case of a homogeneous medium.

This insensitivity of the edge behavior to a discontinuity of medium, in the case of a symmetrical interface was found to hold in the presence of any configuration composed of a zero-thickness screen plus other conductors, provided the symmetry with respect to the interface is maintained. This means that the behavior of the field as the edge of an arbitrary irregular plane screen is approached is the same for a discontinuous medium of the kind described as it is for a continuous one. For a screen which has finite thickness but which is still symmetrical with respect to the interface, the same conclusion was found to hold, at least with respect to conditions in the medium not containing the source excitation. These results follow from an extension of the image procedure of Kelvin.

It is of interest to point out some direct extensions of our results. For instance, suppose we have the same wedge-shaped boundary and the same boundary conditions as before, but we have the equation $\Delta u + k^2 u = 0$ where k is constant. All the results obtained above may be transferred to this situation except that positive powers of r or r_o are replaced by the corresponding Bessel functions, (i.e., $J_\lambda(kr)$) and negative powers of r or r_o are replaced by the corresponding Hankel functions. These Bessel and Hankel functions have the same singularity at the origin as the functions $r^{\pm\lambda}$, and so the qualitative behavior is unaltered. The general qualitative theorems also remain valid as well as the extension of Kelvin's method of images. This is easily seen from the proofs of these theorems, which rely on symmetry considerations and on the fact that the same equation holds in both media.

It is obvious that analysis similar to that which we have given applies to more general boundary conditions and to cases where the potential in question does not necessarily refer to an electrical problem.

Since we are dealing with a local behavior it seems probable that our results as to local behavior remain valid for the case of the equation $\Delta u + k^2(\theta)u = 0$, where $k(\theta)$ is a step function proportional to $\varepsilon(\theta)$, while u obeys the boundary and continuity conditions we have imposed in our work.

At the time this investigation was performed treatment of the electromagnetic problem seemed too difficult. It was hoped that the electrostatic results obtained might constitute an accurate reflection of conditions in the electromagnetic case. (In this connection the magnetostatic field did not come into consideration explicitly, since the magnetic permeability is the same for most materials.) Independently of this work, the particular case of an infinite conducting wedge has since been successfully investigated by Prof. J. Meixner, without the restriction to electrostatics. The reader is referred to this forthcoming work[5].

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